

Momentum space topology in the lattice gauge theory

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Abstract

Momentum space topology of relativistic gauge theory is considered. The topological invariants in momentum space are introduced for the case, when there is the mass gap while the fermion Green functions admit zeros. The index theorem is formulated that relates the number of massless particles and generalized unparticles at the phase transitions to the jumps of the topological invariants. The pattern is illustrated by the lattice model with overlap fermions.

1 Introduction

The main subject of the research presented here is the generalization of topological invariants and index theorem of [1] to the case when the fermion Green function may contain zeros and nonanalytical exceptional points (see also [2]). We expect that models with such unusual properties may be relevant for the description of the new TeV scale physics expected to appear at the LHC. A unified field theory that works somewhere above TeV, may have the Green functions with zeros, our world may live at the position of the phase transition within a model of this kind. The massless fermions and the generalized unparticles appear at the position of this transition. Their numbers are related to the jumps of the topological invariants across the transition.

In the vicinity of non - analytical exceptional point the excitations appear that do not look like ordinary particles. We call them generalized unparticles. This notion is more wide than the notion of usual fermionic unparticles. Namely, the generalized unparticle has the propagator with non - analyticity of general form while in [10, 11] the particular forms of the propagator were derived in accordance with the given scaling dimension d_U .

For the general consideration of momentum space topology see also [14, 15, 16, 17, 18, 19, 20, 21, 22] and [5, 23, 24, 25]. Below momentum space topology of the fermionic system with zeros of the Green function and generalized unparticles is illustrated by the example of the lattice model with overlap fermions [26, 27, 28]. There are critical values of mass parameter $m_0 = 2, 4, 6$ and 8 , at which for $m \neq 0$ the topological quantum phase transitions occur between the insulators with different values of \tilde{N}_4 and \tilde{N}_5 (more on topological phase transitions, at which the topological charge of the vacuum changes while the symmetry does not, see [3]). At these values of m_0 the vacuum states contain the generalized unparticles. At the same time both in these intermediate states and in the insulator states the zeros of the Green function are present in momentum space. The total number of topologically protected unparticles at the transition point is $n_F^u = \Delta\tilde{N}_4$.

For the values of $m_0 \neq 0, 2, 4, 6, 8$ at the phase transition between the states with different signs of m there are no unparticles but massless fermion excitations appear. The number of topologically protected massless fermions n_f^0 is related to the jump of \tilde{N}_5 : $n_f^0 = \Delta\tilde{N}_5/2$.

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2 Green functions without zeros and poles

As well as in [1] we consider the fermion systems with the Green function in Euclidean momentum space of the form

$$\mathcal{G} = \frac{Z[p^2]}{g^i[p]\gamma^i - im[p]}, \quad i = 1, 2, 3, 4 \quad (1)$$

Here $Z(p^2)$ is the wave function renormalization function, while $m(p^2)$ is the effective mass term, $g_a[p]$ are real functions. Z , g , and m may be thought of as the diagonal matrices if several flavors of Dirac fermions are present.

Let us consider the Euclidean Green's function on the 4D lattice \mathcal{G} as the inverse Hamiltonian in 4D momentum space and introduce the 5D Green's function:

$$G^{-1}(p_5, p_4, \mathbf{p}) = p_5 \gamma^5 + \mathcal{G}^{-1}(p_4, \mathbf{p}) = (ip_5 + \mathcal{Q}^{-1}(p_4, \mathbf{p}))(-i\gamma^5). \quad (2)$$

Then one can introduce the topological invariant as the 5-form (see also [19, 20, 21, 22, 29, 30]):

Definition 2.1

$$\mathcal{N}_5 = \frac{1}{2\pi^3 5! i} \text{Tr} \int G dG^{-1} \wedge G dG^{-1} \wedge G dG^{-1} \wedge G dG^{-1} \wedge G dG^{-1}, \quad (3)$$

where the integration is over the Brillouin zone in 4D momentum space (p_4, \mathbf{p}) and over the whole p_5 axis.

The properties of this invariant are summarized in the following lemma:

Lemma 2.2 *Eq. (3) defines the topological invariant for the gapped 4D system with momentum space \mathcal{M} if the following equation holds:*

$$\int_{\partial[\mathcal{M} \otimes R]} \text{tr} ([\delta \mathcal{G}^{-1}] G dG^{-1} \wedge dG \wedge dG^{-1} \wedge dG) = 0, \quad p_5^2 \rightarrow \infty \quad (4)$$

This requirement is satisfied, in particular, for the system with compact closed \mathcal{M} .

For the system with the Green function of the form (1) expression for the topological invariant is reduced to

$$\tilde{\mathcal{N}}_5 = \frac{3}{4\pi^2 4!} \epsilon_{abcde} \int \hat{g}^a d\hat{g}^b \wedge d\hat{g}^c \wedge d\hat{g}^d \wedge d\hat{g}^e, \quad \hat{g}^a = \frac{g^a}{\sqrt{g^e g^e}} \quad (5)$$

(This lemma corresponds to the Theorem from Sect. 4.3 of [1].)

In addition to the invariant $\tilde{\mathcal{N}}_5$ let us also consider a different construction that coincides with $\tilde{\mathcal{N}}_5$ for the case of free fermions

Definition 2.3

$$\tilde{\mathcal{N}}_4 = \frac{1}{48\pi^2} \text{Tr} \gamma^5 \int_{\mathcal{M}} d\mathcal{G}^{-1} \wedge d\mathcal{G} \wedge d\mathcal{G}^{-1} \wedge d\mathcal{G} \quad (6)$$

Here the integration is over the whole 4D space \mathcal{M} .

The expression in this integral is the full derivative. That's why the given invariant can be reduced to the integral over the 3D hypersurface $\partial\mathcal{M}$:

$$\tilde{\mathcal{N}}_4 = \frac{1}{48\pi^2} \text{Tr} \gamma^5 \int_{\partial\mathcal{M}} \mathcal{G}^{-1} d\mathcal{G} \wedge d\mathcal{G}^{-1} \wedge d\mathcal{G} \quad (7)$$

The last equation is identical to that of for the invariant \mathcal{N}_3 for massless fermions. Therefore, Eq. (6) defines the topological invariant if the Green function anticommutes with γ^5 on the boundary of momentum space. In particular, for the noninteracting fermions $\tilde{\mathcal{N}}_4 = \text{Sp } 1$ (the number of Dirac fermions).

The following lemma allows to calculate invariant $\tilde{\mathcal{N}}_4$ in general case:

Lemma 2.4 For the Green function of the form (1) with $\frac{m[p]}{\sqrt{g_a g_a + m^2}} = 0$ ($a = 1, 2, 3, 4$) on $\partial\mathcal{M}$ we have:

$$\tilde{\mathcal{N}}_4 = \frac{1}{2\pi^2 3!} \epsilon_{abcd} \int_{\partial\mathcal{M}} \hat{g}^a d\hat{g}^b \wedge d\hat{g}^c \wedge d\hat{g}^d, \quad \hat{g}^a = \frac{g^a}{\sqrt{g^e g^e}} \quad (8)$$

(This Lemma follows when one substitutes Eq. (1) to Eq. (7).)

Let us introduce the following parametrization

$$\hat{g}_5 = \cos 2\alpha, \quad \hat{g}_a = k_a \sin 2\alpha \quad (9)$$

Vector k may be undefined at the points of momentum space \mathcal{M} , where $\hat{g}^a = 0$, $a = 1, 2, 3, 4$. In nondegenerate case this occurs on points y_i , $i = 1, \dots$. Further we call these points the pseudo - poles of the Green function.

Definition 2.5 The point y_i in momentum space, where $g^a = 0$, $a = 1, 2, 3, 4$ and, therefore, $\mathcal{G}^{-1}[p] = m[p]$, is called pseudo - pole of the Green function.

Actually, in the majority of cases at these points massive fermion excitations appear. This is because in these cases (free continuum fermions, lattice Wilson fermions, overlap fermions, etc) for infinitely small $m[y_i]$ needed to approach continuum limit, the Green function behaves as

$$\mathcal{G} \sim \frac{1}{\lambda \sum_a (-1)^{n_a} q_a \gamma^a - im}, \quad (10)$$

where $q_a = p_a - y_i$, λ is a real constant, n_a are integer constants. Therefore, in Minkowsky space the usual dispersion relation is recovered: $E = \sqrt{q^2 + m^2/\lambda^2}$. We formulate this as the following lemma:

Lemma 2.6 If $g^a \sim \lambda \sum_a (-1)^{n_a} (p^a - y^a)$ while $m \neq 0$ in the small vicinity of $y \in \mathcal{M}$, then at this point massive fermion excitation appears.

Let us denote a small vicinity of the pseudo - pole y_i by $\Omega(y_i)$. According to [1] we have

$$\tilde{\mathcal{N}}_5 = \frac{1}{\pi^2 4!} \epsilon_{abcd} \int_{\sum_{i=0,1,\dots} \partial\Omega(y_i) - \partial\mathcal{M}} (3\hat{g}_5 - \hat{g}_5^3) k^a dk^b \wedge dk^c \wedge dk^d \quad (11)$$

Let us define the 3D analogue of the residue.

Definition 2.7 We denote by $\mathbf{Res}(p)$ the degree of mapping $\{\hat{g}^a : S^3 \rightarrow S^3\}$:

$$\begin{aligned} \mathbf{Res}(p) &= \frac{1}{2\pi^2 3!} \epsilon_{abcd} \int_{\partial\Omega(p)} \hat{g}^a d\hat{g}^b \wedge d\hat{g}^c \wedge d\hat{g}^d, \\ p \in \Omega, \quad |\Omega| \rightarrow 0 \end{aligned} \quad (12)$$

Let us also denote

$$\mathbf{s}(p) = \text{sign } m[p] \quad (13)$$

The lemma follows:

Lemma 2.8 Consider the system with the Green function of the form (1), with $\frac{m[p]}{\sqrt{g_a g_a + m^2}} = 0$ ($a = 1, 2, 3, 4$) on the boundary of momentum space. Let us denote by y_i the points in momentum space, where $g^a = 0$, $a = 1, 2, 3, 4$. Then the topological invariant $\tilde{\mathcal{N}}_5$ is given by

$$\tilde{\mathcal{N}}_5 = \sum_{i=0,1,\dots} \mathbf{s}(y_i) \mathbf{Res}(y_i) \quad (14)$$

(This is Eq. (31) of [1].) For the details of the proofs of the lemmas presented in this section see [1].

3 Green functions with zeros and poles

Here we generalize the definition of the topological invariants to the case, when the Green function may have zeros or poles. Construction of invariants $\tilde{\mathcal{N}}_4$ and $\tilde{\mathcal{N}}_5$ in this case requires some care. The correct definition implies that first we consider momentum space without some vicinities $\Omega(z_i), \Omega(p_i)$ of the points z_i , where \mathcal{G} has zeros and points p_i , where there are poles of the Green function. Then all statements of the previous section (proved in [1]) are valid if we consider the Green function in Momentum space without $\Omega(z_i), \Omega(p_i)$. That's why, say, in Eq. (11) and Eq. (8) of the previous section we must add $-\sum_i \partial\Omega(z_i) - \sum_j \partial\Omega(p_j)$ to $\partial\mathcal{M}$.

Next, the limit is considered when the sizes of these vicinities tend to zero $|\Omega(p_i)|, |\Omega(z_i)| \rightarrow 0$. In order for such a limit to exist the model must obey some requirements.

Remark 3.1 *In this paper we consider only the cases, when exceptional points of the Green function are indeed point - like. We do not consider the situation when exceptional lines or surfaces of the Green function are present. (The case of the exceptional surface may correspond, in particular, to the Fermi surface.)*

For example, if the Green function has the form Eq. (1) and $\hat{g}_5 = \frac{m}{\sqrt{g_a g_a + m^2}} = 0$ ($a = 1, 2, 3, 4$) at z_i, p_i , then the boundaries $\partial\Omega(p_i), \partial\Omega(z_i)$ do not contribute to the sum in (11) at $|\Omega(p_i)|, |\Omega(z_i)| \rightarrow 0$. This means that the mentioned above limit exists for $\tilde{\mathcal{N}}_5$. At the same time under the same conditions $\tilde{\mathcal{N}}_4$ is the topological invariant at $|\Omega(p_i)|, |\Omega(z_i)| \rightarrow 0$ and the points p_i, z_i contribute the sum in Eq. (8).

In order to make formulas more simple let us also introduce the 3D residue at "infinity":

$$\mathbf{Res}(\infty) = -\frac{1}{2\pi^2 3!} \epsilon_{abcd} \int_{\partial\mathcal{M}} \hat{g}^a d\hat{g}^b \wedge d\hat{g}^c \wedge d\hat{g}^d \quad (15)$$

Taking into account Lemma 2.8 and Lemma 2.4, we come to the following

Theorem 3.1 *Suppose the Green function has the form (1) and at its poles and zeros as well as on the boundary of momentum space $\frac{m[p]}{\sqrt{g_a g_a + m^2}} = 0$ ($a = 1, 2, 3, 4$). We denote by y_i the points, where $\mathcal{G}^{-1}(y_i) = m(y_i)$ (pseudo - poles of \mathcal{G}), by z_i the points, where $\mathcal{G}(z_i) = 0$, by p_i the points, where $\mathcal{G}^{-1}(z_i) = 0$. Then $\tilde{\mathcal{N}}_4$ and $\tilde{\mathcal{N}}_5$ are well defined topological invariants. (This means that $\tilde{\mathcal{N}}_4$ and $\tilde{\mathcal{N}}_5$ are not changed under the smooth deformation of \mathcal{G} that keeps the listed above conditions.) As a result we have*

$$\tilde{\mathcal{N}}_5 = \sum_{i=0,1,\dots} \mathbf{s}(y_i) \mathbf{Res}(y_i) \quad (16)$$

and

$$\tilde{\mathcal{N}}_4 = - \sum_{i=0,1,\dots} \mathbf{Res}(z_i) - \sum_{i=0,1,\dots} \mathbf{Res}(p_i) - \mathbf{Res}(\infty) \quad (17)$$

In addition to zeros and poles in general case momentum space may contain non - analytical exceptional points q_i , where \mathcal{G} is not defined but, say, \mathcal{G}^{-1} may differ from zero.

Definition 3.2 *The point q_i in momentum space represents generalized unparticle if in its small vicinity both \mathcal{G} and \mathcal{G}^{-1} are not analytical as functions of momenta.*

As usual, poles p_i of \mathcal{G} (such points that \mathcal{G}^{-1} is zero at p_i but remains analytical in its vicinity) represent massless particles.

Remark 3.3 *If momentum space contains generalized unparticles, then both $\tilde{\mathcal{N}}_5$ and $\tilde{\mathcal{N}}_4$ are not well - defined.*

The pattern of the transition from the state at $\beta > \beta_c$ to the state at $\beta < \beta_c$ can be described in terms of the flow of exceptional points of g^a . Namely, in general there are zeros y_i of g^a , where $\mathcal{G}^{-1} = m$ (we call them pseudo - poles, some of these points become poles $p_i = y_i$ of \mathcal{G} if, in addition, $m[y_i] = 0$). Also there are zeros z_i of \mathcal{G} , where $g^a \rightarrow \infty$. These points cannot simply disappear when the system is changed smoothly with no phase transition encountered. They may annihilate each other if this is allowed by the momentum space topology. Namely, two zeros z_i, z_j may annihilate if $\mathbf{Res}(z_i) + \mathbf{Res}(z_j) = 0$ because in this case they do not contribute the sum in Eq. (17). For the same reason two poles p_i, p_j may annihilate if $\mathbf{Res}(p_i) + \mathbf{Res}(p_j) = 0$. Two pseudo - poles y_i, y_j may annihilate each other if $\mathbf{s}(y_i)\mathbf{Res}(y_i) + \mathbf{s}(y_j)\mathbf{Res}(y_j) = 0$ because in this case they do not contribute the sum in Eq. (16).

Now we are ready to formulate the generalized index theorem:

Theorem 3.2 *Suppose that the 4D system with the Green function of the form (1) depends on parameter β and there is a phase transition at β_c with changing of $\tilde{\mathcal{N}}_4$ and $\tilde{\mathcal{N}}_5$. At $\beta \neq \beta_c$ the system does not contain generalized unparticles and massless excitations. \mathcal{G} as a function of β is smooth everywhere except for the points, where it is not defined (even at the phase transition). Green function may have zeros z_i , and $m[z_i] \neq \infty$. Momentum space \mathcal{M} of the 4D model is supposed to be either compact and closed or open. In the latter case we need that \mathcal{G} does not depend on β on $\partial\mathcal{M}$. Then at $\beta = \beta_c$ the number of topologically protected generalized unparticles is*

$$n_f^u = \Delta\tilde{\mathcal{N}}_4 \quad (18)$$

The number of topologically protected massless excitations at β_c is

$$n_f^0 = \frac{1}{2}\Delta\tilde{\mathcal{N}}_5 - \frac{1}{2}\left\{ \sum_{i:z_j \rightarrow y_i} \mathbf{s}(y_i)\mathbf{Res}(y_i)|_{\beta>\beta_c} - \sum_{i:y_i \rightarrow z_j} \mathbf{s}(y_j)\mathbf{Res}(y_j)|_{\beta<\beta_c} \right\} \quad (19)$$

Here the first sum is over the pseudo - poles that are transformed to zeros at β_c while the second sum is over the zeros that become pseudo - poles.

The **Proof** is given in [2].

Remark 3.4 *There are two important particular cases:*

1. *If zeros are not transformed to pseudo - poles and vice versa, then*

$$n_f^0 = \frac{1}{2}\Delta\tilde{\mathcal{N}}_5 \quad (20)$$

2. *If $\text{sign}(m)$ remains constant on \mathcal{M} and as a function of β , then $n_f^0 = 0$ and*

$$n_f^0 = \frac{1}{2}\Delta\{\tilde{\mathcal{N}}_5 - \mathbf{s}\tilde{\mathcal{N}}_4\} = 0 \quad (21)$$

From here we obtain $\Delta\tilde{\mathcal{N}}_5 = \text{sign}(m)\Delta\tilde{\mathcal{N}}_4$ if $\text{sign}(m) = \text{const}$. In the other words,

$$\Delta\left\{ \sum_{i=0,1,\dots} \mathbf{Res}(z_i) + \sum_{i=0,1,\dots} \mathbf{Res}(y_i) \right\} = 0 \quad (22)$$

Remark 3.5 *The total observed number of generalized unparticles and massless fermions may be larger than n_f^u and n_f^0 . In this case some of these excitations may annihilate each other without a phase transition.*

In a similar way the following theorem may be proved

m	$\tilde{\mathcal{N}}_4$	$\tilde{\mathcal{N}}_5$
$m > 0$	0	0
$-2 < m < 0$	0	-2
$-4 < m < -2$	0	6
$-6 < m < -4$	0	-6
$-8 < m < -6$	0	2
$m < -8$	0	0

Table 1: The values of topological invariants $\tilde{\mathcal{N}}_4$ and $\tilde{\mathcal{N}}_5$ for free Wilson fermions.

Theorem 3.3 Suppose that the 4D system with the Green function of the form (1) depends on parameter β and there is a phase transition at β_c with changing of $\tilde{\mathcal{N}}_4$ and $\tilde{\mathcal{N}}_5$. At $\beta \neq \beta_c$ all excitations are massless. \mathcal{G} as a function of β is smooth everywhere except for the points, where it is not defined. Green function may have zeros z_i , and $m[z_i] \neq \infty$. Momentum space \mathcal{M} of the 4D model is supposed to be either compact and closed or open. In the latter case we need that \mathcal{G} does not depend on β on $\partial\mathcal{M}$. Then at $\beta = \beta_c$ the number of topologically protected generalized unparticles is

$$n_f^u = \Delta\tilde{\mathcal{N}}_4 \quad (23)$$

4 An example: overlap fermions

In lattice regularization the periodic boundary conditions are used in space direction and antiperiodic boundary conditions are used in the imaginary time direction. The momenta to be considered, therefore, also belong to a lattice:

$$p_a = \frac{2\pi K_a}{N_a} \quad p_4 = \frac{2\pi K_4 + \pi}{N_t}, \quad K_a, K_4 \in Z \quad a = 1, 2, 3 \quad (24)$$

Here N_a, N_t are the lattice sizes in x, y, z , and imaginary time directions, correspondingly.

For the free Wilson fermions the Green function has the form [32]:

$$\begin{aligned} \mathcal{G} &= \left(\sum_a \gamma_a \sin p_a - i(m + \sum_a (1 - \cos p_a)) \right)^{-1} \\ &= \frac{\sum_a \gamma_a \sin p_a + i(m + \sum_a (1 - \cos p_a))}{\sum_a \sin^2 p_a + (m + \sum_a (1 - \cos p_a))^2}, \quad a = 1, 2, 3, 4 \end{aligned} \quad (25)$$

In Table 1 the values of $\tilde{\mathcal{N}}_5$ for the Wilson fermions calculated in [1] are presented. The index theorem states that the total number n_F of gapless fermions emerging at the critical values of mass m is determined by the jump in $\tilde{\mathcal{N}}_5$:

$$n_F = \frac{1}{2} \Delta\tilde{\mathcal{N}}_5 \quad (26)$$

It is worth mentioning that $\tilde{\mathcal{N}}_4 = 0$ for Wilson fermions.

When the interaction of Wilson fermions with the lattice gauge field $\mathcal{U} = e^{i\mathcal{A}}$ defined on links is turned on, we have (in coordinate space):

$$\mathcal{G}(x, y) = \frac{i}{Z} \int D\mathcal{U} \exp(-S_G[\mathcal{U}]) \text{Det}(\mathcal{D}[\mathcal{U}, m]) \mathcal{D}_{x,y}^{-1}[\mathcal{U}, m] \quad (27)$$

where S_G is the gauge field action while

$$\mathcal{D}_{x,y}[\mathcal{U}, m] = -\frac{1}{2} \sum_i [(1 + \gamma^i) \delta_{x+\mathbf{e}_i, y} \mathcal{U}_{x+\mathbf{e}_i, y} + (1 - \gamma^i) \delta_{x-\mathbf{e}_i, y} \mathcal{U}_{x-\mathbf{e}_i, y}] + (m + 4) \delta_{xy} \quad (28)$$

$m_0 < 0$	-	-	-	-	-
$2 > m_0 > 0$	$1 \otimes m$	-	-	-	-
$4 > m_0 > 2$	$1 \otimes m$	$4 \otimes [m(1 - \frac{2}{m_0})]$	-	-	-
$6 > m_0 > 4$	$1 \otimes m$	$4 \otimes [m(1 - \frac{2}{m_0})]$	$6 \otimes [m(1 - \frac{4}{m_0})]$	-	-
$8 > m_0 > 6$	$1 \otimes m$	$4 \otimes [m(1 - \frac{2}{m_0})]$	$6 \otimes [m(1 - \frac{4}{m_0})]$	$4 \otimes [m(1 - \frac{6}{m_0})]$	-
$m_0 > 8$	$1 \otimes m$	$4 \otimes [m(1 - \frac{2}{m_0})]$	$6 \otimes [m(1 - \frac{4}{m_0})]$	$4 \otimes [m(1 - \frac{6}{m_0})]$	$1 \otimes [m(1 - \frac{8}{m_0})]$

Table 2: The spectrum of the system with free overlap fermions. In the first column the values of m_0 are specified. In the other columns masses of the doublers are listed. Expression $x \otimes v$ means x states with the masses equal to v .

Here \mathbf{e}_i is the unity vector in the i -th direction. Again, the Green function in momentum space is expected to have the form [28] of Eq. (1).

Let us consider briefly the properties of overlap fermions. In this regularization the propagator has the form:

$$\mathcal{G}(x, y) = \frac{1}{Z} \int D\mathcal{U} \exp\left(-\tilde{S}_G[\mathcal{U}]\right) \{-i\mathbf{D}[\mathcal{U}] - im\}_{x,y}^{-1} \quad (29)$$

Here the effective action \tilde{S}_G includes also the fermion determinant, the overlap operator is defined as

$$\mathbf{D} = \frac{2m_0}{\mathcal{O}^{-1} - 1} \quad (30)$$

with

$$\mathcal{O}_{x,y}[\mathcal{U}] = \frac{1}{2} \left(1 + \frac{\mathcal{D}[\mathcal{U}, -m_0]}{\sqrt{\mathcal{D}^+[\mathcal{U}, -m_0] \mathcal{D}[\mathcal{U}, -m_0]}} \right) \quad (31)$$

Here m_0 and m are bare mass parameters [27]. The parameter m represents bare physical mass.

In spite of a rather complicated form of the expression for the overlap operator it is commonly believed that the Green function in momentum space has the same form (1) as the Green function for Wilson fermions [27]. In particular, for the free overlap fermions the Green function has the form:

$$\mathcal{G}(p) = \frac{1}{g^i[p] \gamma^i - im} \quad (32)$$

with (see Appendix in [27]):

$$\begin{aligned} g^i[p] &= 2m_0 \sin p^i \frac{A(p) + \sqrt{A(p)^2 + \sum_i \sin^2 p^i}}{\sum_i \sin^2 p^i}, \\ A(p) &= -m_0 + \sum_i [1 - \cos p^i] \end{aligned} \quad (33)$$

In Table 2 we represent the spectrum of the model for different values of m_0 . For $0 < m_0 < 2$ one obtains that \hat{g}^a have the only zero at $p = 0$. However, there are poles of \hat{g}^a at $p_{n_i} = (\pi n_1, \pi n_2, \pi n_3, \pi n_4)$, $n_i = 0, 1$, $\sum n^2 \neq 0$. The values of \hat{g}_5 at these points vanish.

In general case some of the poles of \hat{g}^a may become zeros depending on the value of m_0 . For positive $A(p_{n_i})$ we obtain $g^i[p_{n_i} + \delta p] \sim 4m_0 A(p_{n_i}) (-1)^{n_i} \frac{\delta p^i}{|\delta p|^2}$, where $|\delta p|^2 = \sum_i [\delta p^i]^2$ and

$$\mathcal{G}(p_{n_i} + \delta p) \sim \frac{\frac{1}{4m_0 A(p_{n_i})} |\delta p|^2}{\sum_i (-1)^{n_i} \delta p^i \gamma^i - i \frac{m}{4m_0 A(p_{n_i})} |\delta p|^2} \sim \frac{1}{4m_0 A(p_{n_i})} \sum_i (-1)^{n_i} \delta p^i \gamma^i \quad (34)$$

We have zeros of the Green function at these points.

For negative $A(p_{n_i})$ we obtain $g^i[p_{n_i} + \delta p] \sim \frac{m_0}{|A(p_{n_i})|}(-1)^{n_i} \delta p^i$ and

$$\mathcal{G}(p_{n_i} + \delta p) \sim \frac{\frac{|A(p_{n_i})|}{m_0}}{\sum_i (-1)^{n_i} \delta p^i \gamma^i - im \frac{|A(p_{n_i})|}{m_0}} \quad (35)$$

The values $A(p_{n_i}) = -m_0 + 2 \sum_i n_i$ at these points are related to the values of the masses of the doublers: $\mathbf{m}_{n_i} = m(1 - \frac{2}{m_0} \sum_i n_i)$.

The special situation appears if $A(p_{n_i}) = 0$ (this occurs for the intermediate values $m_0 = 2, 4, 6, 8$):

$$\mathcal{G}(p_{n_i} + \delta p) \sim \frac{\frac{1}{2m_0}}{\sum_i (-1)^{n_i} \frac{\delta p^i}{|\delta p|} \gamma^i - i \frac{m}{2m_0}}, \quad |\delta p| = \sqrt{\sum_i [\delta p^i]^2} \quad (36)$$

In this case the Green function is not defined at p_{n_i} and the unparticles appear with the propagator equal (up to the normalization constant) to that of presented in [10] (follows from Eq. (8) of [10] with $\alpha = \beta = 0, \zeta \neq 0, d_U = 2$). At $m = 0$ we arrive at the propagator given in Eq. (10) of [10] with $\alpha = 0, d_U = 2$.

It is worth mentioning that for the overlap fermions unlike the Wilson fermions there are zeros of the Green function at some points in momentum space (see Eq. (34)). Moreover, at the intermediate values of m_0 the Green function is undefined at some points (see Eq. (36)). As a result, we need to consider momentum space without small vicinities of both mentioned types of the points in order to calculate $\tilde{\mathcal{N}}_5$ and $\tilde{\mathcal{N}}_4$. Momentum space, therefore, becomes open. At $m_0 \neq 0, 2, 4, 6, 8$ we have at the points, where \mathcal{G} has zeros $\hat{g}_5 = 0$ due to Eq. (34). Therefore, the conditions of Theorem 3.1 are satisfied. For the intermediate values of m_0 this theorem cannot be applied.

The values of $\tilde{\mathcal{N}}_4$ and $\tilde{\mathcal{N}}_5$ for overlap fermions versus parameter m_0 are represented in Table 3. Let us remind that unlike the free Wilson fermions at intermediate values of the mass parameter $m_0 = 0, 2, 4, 6, 8$ there exist exceptional points in momentum space such that the Green function is undefined at these points. Due to this the invariants $\tilde{\mathcal{N}}_4$ and $\tilde{\mathcal{N}}_5$ are not well defined in the intermediate states.

When the interaction with the gauge fields is turned on, it is necessary to check that $\tilde{\mathcal{N}}_4$ and $\tilde{\mathcal{N}}_5$ remain topological invariants. Our check shows that both expressions remain the topological invariants.

In the intermediate states at $m_0 = 2, 4, 6, 8, m \neq 0$ there are no true massless states. Using data of Table 3 one finds that this is in accordance with Eq. (21) of the index theorem. However, there are the generalized unparticles with the Green function given by Eq. (36) (see the definition in Section 3). The number of generalized unparticles is related to the jump in $\tilde{\mathcal{N}}_4$:

$$n_f^u = \Delta \tilde{\mathcal{N}}_4 \quad (37)$$

This relation can easily be checked and is also in accordance with theorem 3.2.

In the intermediate state with $m = 0, m_0 \neq 0, 2, 4, 5, 8$ the generalized unparticles are absent. Zeros of \mathcal{G} remain zeros across the transition. However, there are massless fermions. Their number is: 0 for $m_0 < 0$, 1 for $2 > m_0 > 0$, 5 for $4 > m_0 > 2$, 11 for $6 > m_0 > 4$, 15 for $8 > m_0 > 6$, 16 for $m_0 > 8$. At the same time the number of topologically protected massless fermions given by Eq. (20) is:

$$n_f = \frac{1}{2} \Delta \tilde{\mathcal{N}}_5 \quad (38)$$

This is 0 for $m_0 < 0$, 1 for $2 > m_0 > 0$, -3 for $4 > m_0 > 2$, 3 for $6 > m_0 > 4$, 1 for $8 > m_0 > 6$, 0 for $m_0 > 8$. Therefore, except for the conventional case $2 > m_0 > 0$ there are massless fermions in the intermediate states that are not protected by momentum space topology. When the interaction with the gauge field is turned on some of them may annihilate each other so that the total number of massless fermions is reduced without the phase transition.

m_0	$\tilde{\mathcal{N}}_4$	$\tilde{\mathcal{N}}_5$
$-m_0 > 0$	0	0
$-2 < -m_0 < 0$	1	$\text{sign } m$
$-4 < -m_0 < -2$	-3	$-3 \text{sign } m$
$-6 < -m_0 < -4$	3	$3 \text{sign } m$
$-8 < -m_0 < -6$	-1	$-\text{sign } m$
$-m_0 < -8$	0	0

Table 3: The values of topological invariants $\tilde{\mathcal{N}}_4$ and $\tilde{\mathcal{N}}_5$ for free overlap fermions.

There are also mixed intermediate states $m = 0, m_0 = 2, 4, 6, 8$, where both massless fermions and generalized unparticles are present. The corresponding transitions satisfy the conditions of Theorem 3.3. All pseudo - poles of the Green function become true poles. The zeros of the Green function may be transformed to the massless excitations across the transition points at $m = 2, 4, 6, 8$. At the corresponding points the generalized unparticles appear. Their number is equal to the jump of $\tilde{\mathcal{N}}_4$. The corresponding values are listed in Table 3.

5 Discussion

The topological invariants do not feel smooth changes of the model. Only a phase transition may lead to the change of the topological invariant. Therefore, if the free system (without gauge fields) and the interacting system (with gauge fields) belong to the same phase, then the values of the topological invariants are the same. So, we may calculate the topological invariant for the free fermions and it will be equal to the same value for the complicated interacting system that is related to the free system by a smooth transformation.

The vacuum states of lattice models with fully gapped fermions but with zeros in Green function (insulating vacua) in 4D space-time are characterized by two topological invariants, $\tilde{\mathcal{N}}_4$ and $\tilde{\mathcal{N}}_5$. They are responsible for the number of generalized unparticles and gapless fermions which appear at the topological transitions between the massive states with different topological charges.

The continuum limit of the model with overlap fermions considered in Section 4 at $m = 0, m_0 = 2, 4, 6, 8$ may be taken seriously. In such limit a continuum theory appears that contains the unparticle excitations. At the same time, the general properties of the quantum phase transition with change of $\tilde{\mathcal{N}}_4, \tilde{\mathcal{N}}_5$ can be applied to the relativistic field theories with fermions. So, we may have the new look at the high energy field theoretical models. The entireties, that are new for the high energy physics, appear. These are the zeros and the non - analytical exceptional points of the Green function. We relate the latter points to the generalized unparticles.

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